Exercise 65

Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

$$x^2 + y^2 = r^2, \quad ax + by = 0$$

Solution

The points of intersection are found by solving the system of equations for x and y.

$$\begin{cases} x^2 + y^2 = r^2 \\ ax + by = 0 \end{cases}$$

Solve the second equation for y.

$$y = -\frac{a}{b}x$$

Differentiate both sides of the given equations with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(r^2) \qquad \frac{d}{dx}(ax + by) = \frac{d}{dx}(0)$$

Use the chain rule to differentiate $y = y(x)$.

$$2x + 2y\frac{dy}{dx} = 0 \qquad a + b\frac{dy}{dx} = 0$$

Solve each equation for dy/dx.

$$2y\frac{dy}{dx} = -2x \qquad b\frac{dy}{dx} = -a$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \frac{dy}{dx} = -\frac{a}{b}$$

At any point of intersection $y = -\frac{a}{b}x$, so the slopes of the tangent lines are as follows.

$$\frac{dy}{dx} = -\frac{x}{-\frac{a}{b}x} \qquad \frac{dy}{dx} = -\frac{a}{b}$$

$$\frac{dy}{dx} = \frac{b}{a} \qquad \frac{dy}{dx} = -\frac{a}{b}$$

The slopes are negative reciprocals at the points of intersection; therefore, the familes of curves defined by $x^2 + y^2 = r^2$ and $ax + by = 0$ are orthogonal trajectories.

Observe that at all points of intersection the tangent lines to the members of each family of curves are orthogonal.